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Simultaneous Approximation of a Set

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A family in a linear space is to be simultaneously approximated by a finite-dimensional linear subspace with respect to a norm. Existence of a best approximation is established. Characterization and uniqueness results are obtained. Discretization of the family can be employed to get a best approximation.

Let \mathcal{F} be a linear space with norm $\|\cdot\|$. Let $\{\phi_1, \dots, \phi_n\}$ be a linearly independent subset of \mathcal{F} and define

$$L(A) = \sum_{k=1}^n a_k \phi_k.$$

The problem of approximation is: For a given nonempty subset F of \mathcal{F} , find a parameter A to minimize

$$e(F, A) = \sup\{\|f - L(A)\| : f \in F\}.$$

Such a parameter A is called best and $L(A)$ is called a best approximation to F . This is a generalization of the classical problem of linear approximation with respect to a norm, in which F consists of a single element f .

The omitted proofs of results in this paper use arguments similar to the classical ones for the case of F a single element.

One of the first to consider this general approximation problem was Golomb [7, pp. 264ff]. It has also been considered by Laurent and Tuan [8] and for F a pair by Goel *et al.* [6]. A related approximation problem is that of approximation by all of \mathcal{F} , the problem of centers [9].

A case of special interest has been the case in which $\|\cdot\|$ is the Chebyshev norm. The first widely distributed result on this was by the author [5]. Laurent and Tuan cite two neglected earlier papers by Golomb and Remez. Diaz and McLaughlin extended the author's results in [2] and studied complex linear Chebyshev approximation in [3]. Nonlinear simultaneous Chebyshev approximation has been studied by Blatt [1].

The problem of simultaneous approximation arises in approximation of a function (weakly) dependent on a parameter or parameters. For example, let h be a function on $X \rightarrow y$. Let us define

$$g_y(x) = h(x, y), \quad x \in X, \quad y \in Y,$$

and set $F = \{g_y; y \in Y\}$. In the case that h is measured experimentally and the value of the parameter y is difficult to measure, we may know the family F much better than h . In the case h is approximated by a function of x only, it seems appropriate to approximate the family F simultaneously. In the case that the value of the parameter when the measurement was taken is unknown, we have effectively a multivalued function g of x , and simultaneous approximation is the only possible approach.

This approach is particularly suitable for Chebyshev approximation of a function of several variables by functions of fewer variables. Suppose we wish to approximate h , a function of $X \rightarrow Y$ by a function of X . Let s be an approximant and F be as in the preceding paragraph, then

$$\|h - s\|_\infty = \sup\{\|f - s\|_\infty; f \in F\}.$$

Motivation for studying this problem is also given in [7, pp. 264 ff; 8, p. 138; 9, p. 283].

It can be shown that $e(F, A) = e(\bar{F}, A)$, where \bar{F} is the closure of F . Hence there is no loss of generality in considering F closed.

THEOREM. *Let F be a bounded nonempty set, then there exists a best approximation to F .*

THEOREM. *The set of best coefficient vectors is closed, bounded, and convex.*

COROLLARY. *$e(F, \cdot)$ is a convex function.*

Use of a routine for minimizing a convex function is a general approach to computing best approximations. In some cases this is the only known approach.

Let $M(F, A) = \{f; \|f - L(A)\| = e(F, A)\}$.

THEOREM. *Let F be compact. A necessary and sufficient condition for A to be best to F is that A is best to $M(F, A)$.*

Proof. Sufficiency is obvious. Suppose A is not best to $M(F, A)$. Then there is a coefficient vector B such that $e(M(F, A), B) < e(M(F, A), A)$. Hence

$$\|f - L(B)\| < \|f - L(A)\|, \quad f \in M(F, A).$$

By continuity there is an open cover U of $M(F, A)$ such that

- (i) $\|f - L(B)\| < \|f - L(A)\|, \quad f \in U$
- (ii) for $0 < \lambda \leq 1$ and such f ,

$$\begin{aligned} \|f - L(\lambda B + (1 - \lambda)A)\| &\leq \lambda \|f - L(B)\| + (1 - \lambda) \|f - L(A)\| \\ &< \|f - L(A)\|. \end{aligned} \quad (1)$$

If U contains all of F , A is not best to F . Otherwise let $V = F \sim U$ be nonempty. As U is open, V is a closed set. Let

$$\eta = \sup\{\|f - L(A)\| : f \in V\},$$

then as V is compact, there is $f \in V$ for which the supremum is attained and since $M(F, A) \cap V$ is empty, $\eta < e(F, A)$. There exists $\mu > 0$ such that $\mu \|L(B - A)\| < e(F, A) - \eta$. We have for $f \in V$, $0 < \lambda \leq \mu$,

$$\begin{aligned} \|f - L(\lambda B + (1 - \lambda)A)\| &< \|f - L(A)\| + \|\lambda L(B - A)\| \\ &< \eta + e(F, A) - \eta = e(F, A). \end{aligned}$$

Combining this with (1) we have for all λ in $(0, \min\{\mu, 1\}]$

$$\|f - L(\lambda B + (1 - \lambda)A)\| < e(F, A), \quad f \in F,$$

and by compactness, $e(F, \lambda B + (1 - \lambda)A) < e(F, A)$, proving the theorem.

It is a consequence of the theorem that if $M(F, A)$ consists of a single element f and $L(A)$ is a unique best approximation to f , $L(A)$ is a unique best approximation to F .

THEOREM. *Let $\|\cdot\|$ be strict and F compact. Then a best approximation is unique.*

Let us now consider some special norms. The L_p norms are strict for $1 < p < \infty$ and are covered by the above theorem. The Chebyshev norm is not strict and nonuniqueness can occur. The papers [1; 5] should be consulted. In the case of the L_1 norm, it appears that nonuniqueness can occur easily.

THEOREM. *Let \int be the integral on X . Let $F = \{f_1, f_2\}$, $f_1 \leq f_2$. If $f_1 \leq L(A) \leq f_2$, and $\int(L(A) - f_1) = \int(f_2 - L(A)) = \int(f_2 - f_1)/2$, then $L(A)$ is a best L_1 approximation to F .*

In the case where f_1 and f_2 are different constants and L is a polynomial approximating function of degree ≥ 1 , there are many such A 's. Carroll [4]

gives a sufficient condition for best simultaneous L_1 approximations to be unique.

Let

$$d(F_1, F_2) = \sup\{\inf\{\|f_1 - f_2\| : f_2 \in F_2\} : f_1 \in F_1\},$$

$$\text{dist}(F_1, F_2) = \max\{d(F_1, F_2), d(F_2, F_1)\}.$$

We say $\{F_k\} \rightarrow F$ if $\text{dist}(F, F_k) \rightarrow 0$.

THEOREM. *Let $\{F_k\} \rightarrow F$ and A^k be best to F_k . Then $\{A^k\}$ has an accumulation point and any such accumulation point is best to F .*

A case of particular interest is where F_k is a sequence of subsets of F such that for all $f \in F$, there is a sequence $\{f_k\} \rightarrow f, f_k \in F_k$. In this case it is seen that $\{F_k\} \rightarrow F$ and the theorem can be applied. The case where $\{F_k\}$ is a sequence of finite sets is of practical interest, as computing a best approximation relative to a set F of functions is easier if F is finite. The process of replacing approximation of infinite F by approximation of finite F_k is a process of *discretization*, which is used extensively in many other approximation problems.

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